Construction of Euclidean Tessellations with MAPLE and Classpad300

The regular tessellation of the euclidean plane is a covering of the plane without gaps and overlappings, by polygons all of which have the same size and shape. Each individual polygon will be called a tile. If $m, n \in \mathbb{N}$, with $\frac{1}{m}+\frac{1}{n}=\frac{1}{2}$, then there is a regular tessellation of the euclidean plane by regular n-gons so that m such n-gons meet at every vertex. We remark that in the hyperbolic and spherical cases the corresponding conditions are $\frac{1}{m}+\frac{1}{n}<\frac{1}{2}$ and $\frac{1}{m}+\frac{1}{n}>\frac{1}{2}$ respectively.

In the Euclidean case the former condition impose the existence of only three type of tilings where $(m,n)=(4,4),(3,6)$ and $(6,3)$.

The simplest, and probably the earliest type of tessellation is $(4,4)$ the one which use a square as tile and in which the squares are arranged in the same manner as on the chess board.

The concepts of congruent tiles implies the existence of the group of self mappings of the Euclidean plane (namely the group of rigid motions), which allows us to define polygons of the same shape and size in different parts of the plane.

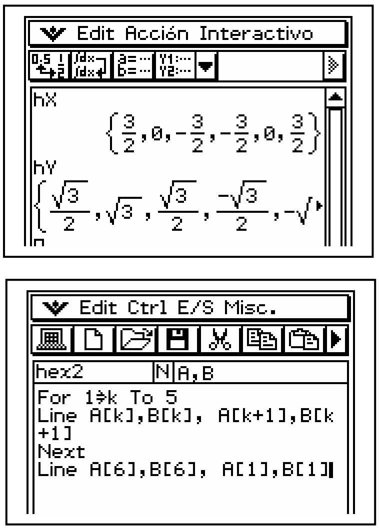
Example 1.

The tessellation (3,6) that use hexagons, can be built by means of a group that possesses two generators: a horizontal translation $gT=(3,0)$ and a diagonal translation $gS=\left(\frac{3}{2},\frac{3}{2}\sqrt{3}\right)$. In the box, below, the program "**hex3**" for the calculator Casio Classpad300 is shown. This program allows to build a tessellation with hexagons, using a group with the generators “gT” and “gS”.

The program uses the subroutine "**hex2**" that is shown in the figure 1. This subroutine allows to draw a hexagon given two lists which contain the x-values and y-values respectively of the coordinates of the points that form the vertexes of the hexagon. The initial hexagon is built starting from the lists "hX" and "hY" and it is given in the figure 2.

```
DelVar hX,hY,gT,gS,tran,i,j,k
{3/2,0,-3/2,-3/2,0,3/2}⇒ hX
{√3/2 ,√3 ,√3/2 ,−√3/2 ,−√3 ,−√3/2 }⇒hY
{3,0}⇒gT
{3/2, 3/2×√3 }⇒gS
ViewWindow -10,10,1,-5,5,1
For -2⇒i To 2
For -2⇒j To 2
i×gT+j×gS⇒tran
hex2(hX+tran[1],hY+tran[2])
Next
Next
```

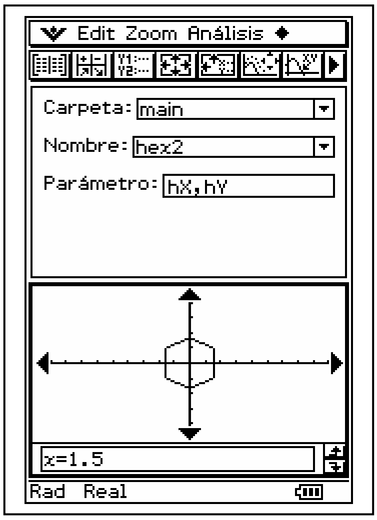
Program “**hex3**”



```
hX
{3/2,0,-3/2,-3/2,0,3/2}
hY
{√3/2,√3,√3/2,-√3/2,-√
```

```
hex2        N A,B
For 1⇒k To 5
Line A[k],B[k], A[k+1],B[k+1]
Next
Line A[6],B[6], A[1],B[1]
```

Figure1



Carpeta: main
Nombre: hex2
Parámetro: hX,hY
x=1.5
Rad Real

Figure 2

The result of applying the program "**hex3**", which is shown in the figure 3, is a tessellation formed by hexagons that is invariant under the action of the group of transformations obtained by means of the composition of the generators “gT” and “gS”.

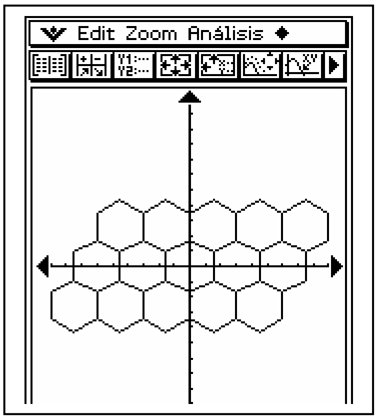


Figure 3

Example 2

Let us consider now, the same tessellation formed by hexagons, but this time obtained by means of the action of a group that possesses three generators: a horizontal translation $gT = (3,0)$, a diagonal translation $gS = \left(\frac{3}{2}, \frac{3}{2}\sqrt{3}\right)$ and a rotation

$$gR = \begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{pmatrix} \text{ with the angle } \frac{\pi}{3}.$$

The program "**hex4**" shown in the following box considers the composition of these three generators for the construction of a tessellation with hexagons.

```
DelVar hX,hY,gT,gS,tran,i,j,k,s
{3/2,0,-3/2,-3/2,0,3/2}⇒ hX
{√3/2 ,√3 ,√3/2 ,–√3/2 ,–√3 ,–√3/2 }⇒hY
{3,0}⇒gT
{3/2,3/2×√3 }⇒gS
ViewWindow -30,30,1,-15,15,1
For 0⇒j To l
For 0⇒ s To p
For 0⇒ i To m
i×gT+j×gS⇒tran
cos(s×π/3)×(hX+tran[1])-sin(s×π/3)×(hY+tran[2])⇒nX
sin(s×π/3)×(hX+tran[1])+cos(s×π/3)×(hY+tran[2])⇒nY
hex2(nX,nY)
Next
Next
Next
```

Program "**hex4**"

In Maple a similar program can be built, using for it the procedure **RTS** that is shown in the following box.

```
> RTS:=proc(k,l,a)
local hexX,hexY,genT,genS,M,trans,newhexX,newhexY:
hexX:=[3/2,0,-3/2,-3/2,0,3/2]:
hexY:=[sqrt(3)/2,sqrt(3),sqrt(3)/2,-sqrt(3)/2,-sqrt(3),-sqrt(3)/2]:
genT:=[3,0]:
genS:=[3/2,3/2*sqrt(3)]:
M:=<<cos(a) |-sin(a)> , <sin(a) | cos(a)>>:
trans:=k*genT+l*genS:
newhexX:=map(x->x+trans[1],hexX):
newhexY:=map(x->x+trans[2],hexY):
plots[polygonplot](zip((x,y)-
>linalg[multiply](M,[x,y]),newhexX,newhexY),scaling=constrained,axes=none,color=COL
R(RGB,rand()/10^12,rand()/10^12,rand()/10^12));
end proc:
```

Procedure "**RTS**"

For example, when applying the following command will be obtained the original hexagon and another hexagon horizontally translated and later rotated by an angle $\pi/3$ (figure 4)

```
> plots[display](RTS(0,0,0),RTS(1,0,Pi/3));
```

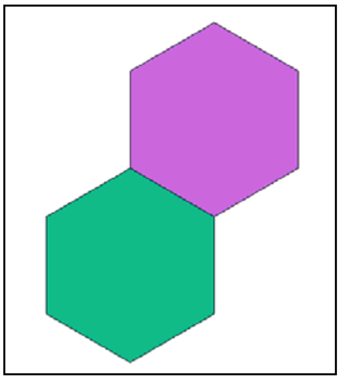


Figure 4

The program "**hex4**" contains the parameters "m", "l" and "p" that indicate how many times each generator is applied in the transformation "$T^i S^j R^s$" with $0 \le i \le m$, $0 \le j \le l$ and $0 \le s \le p$. In general, we can consider negative values for these parameters. Negative values indicate the inverse transformations. For specific relationships among these parameters interesting figures take place as those shown in the next figures. The letters (a) were made with maple and the letters (b) with classpad300.

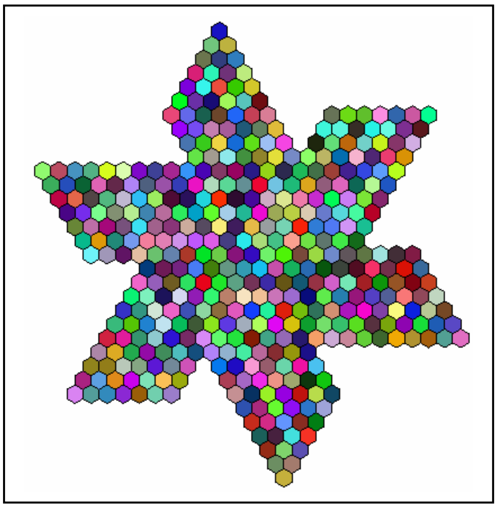


Figure 5(a)

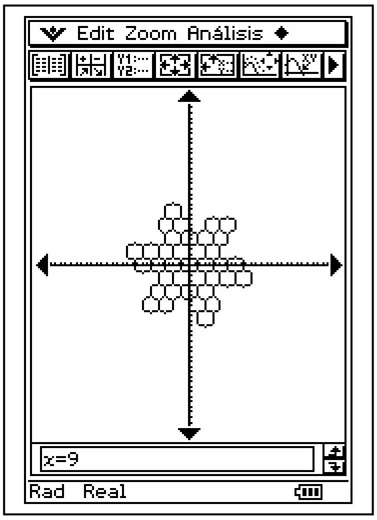


Figure 5(b) $m = 1, l = 3, p = 6$

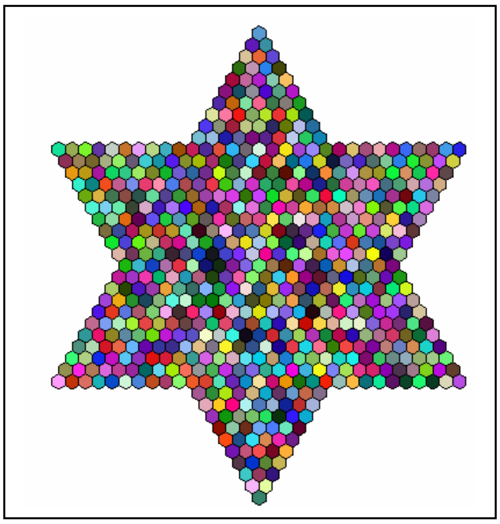
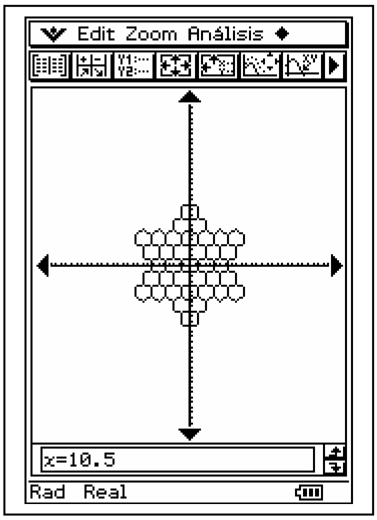


Figure 6(a)



x=10.5

Figure 6(b)

$m = 2, l = 2, p = 6$

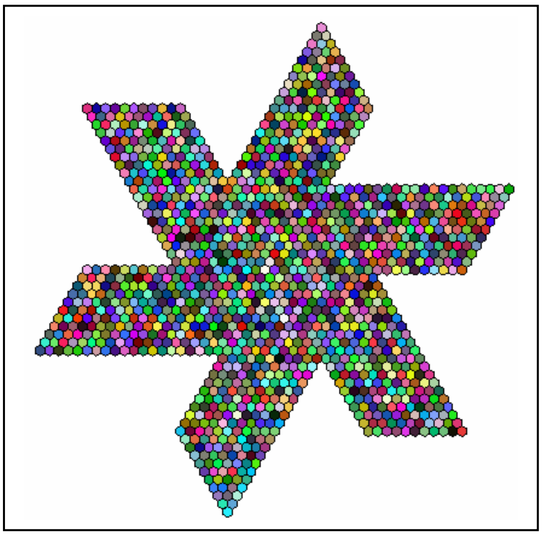
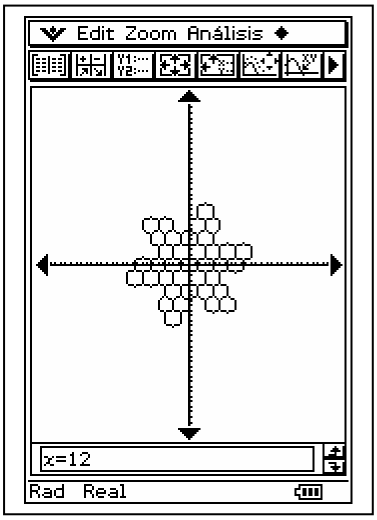


Figure 7(a)



x=12

Figure 7(b)

$m = 3, l = 1, p = 6$

The repetition or iteration of the generator "gT", "gS " and "gR " will lead to fill the whole plane, as their “final attractor", however in the process interesting symmetries can be observed as those shown in the figures 5 – 7, which appear as a result of the increment in the complexity and the repeated action of the generators.

The following maple commands produce the figure (5a), (6a) and (7a) respectively:

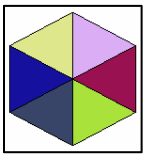
```
>plots[display](seq(seq(seq(RTS(n,i,j*Pi/3),i=0..10),j=0.
.6),n=0..6));
```

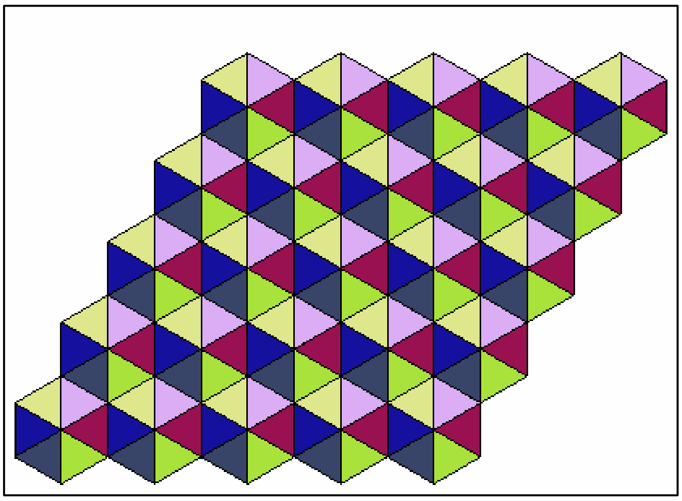
```
>plots[display](seq(seq(seq(RTS(n,i,j*Pi/3),i=0..10),j=0.
.6),n=0..10));
```

```
>plots[display](seq(seq(seq(RTS(n,i,j*Pi/3),i=0..10),j=0.
.6),n=0..20));
```

Example 3

```
> K:=2: L:=2:
> TriX:=[0,3/2,3/2]:
TriY:=[0,sqrt(3)/2,-sqrt(3)/2]:
> for r from 0 to 6 do
M:=<<cos(Pi/3*r) |-sin(Pi/3*r)> , <sin(Pi/3*r) |
cos(Pi/3*r)>>:
p[r]:=plots[polygonplot](zip((x,y)-
>linalg[multiply](M,[x,y]),TriX,TriY),color=COLOR(RGB,ran
d()/10^12,rand()/10^12,rand()/10^12));
od:
> h:=plots[display](seq(p[i],i=1..6),axes=none):
> h;
```





```
> genT:=[3,0]:
genS:=[3/2,3/2*sqrt(3)]:
for k from -K to K do
for l from -L to L do
trans:=k*genT+l*genS:
p[k,l]:=plottools[translate](h,trans[1],trans[2]);
od:
od:
> plots[display](seq(seq(p[i,j],i=-K..K),j=-L..L),
axes=none, scaling=constrained);
```

References

[1] D. Mumford, Caroline Series, David Wright, “Indra’s Pearls, The vision of Felix Klein”.